ISSN 2072-5981
doi: 10.26907/mrsej

Magnetic Resonance in Solids

Electronic Journal

Volume ***26***
Issue ***1***
Article No ***24105***
1-14 pages
April, ***19***
2024
doi: **10.26907/mrsej-24105**

http://mrsej.kpfu.ru
https://mrsej.elpub.ru

Established and published by Kazan University*
Endorsed by International Society of Magnetic Resonance (ISMAR)
Registered by Russian Federation Committee on Press (#015140),
August 2, 1996
First Issue appeared on July 25, 1997

"***Magnetic Resonance in Solids. Electronic Journal***" (**MRS*ej***) is a peer-reviewed, all electronic journal, publishing articles which meet the highest standards of scientific quality in the field of basic research of a magnetic resonance in solids and related phenomena.

Indexed and abstracted by
Web of Science (*ESCI*, Clarivate Analytics, from 2015),
Scopus (Elsevier, from 2012), *RusIndexSC* (*eLibrary*, from 2006), *Google Scholar*,
DOAJ, *ROAD*, *CyberLeninka* (from 2006), *SCImago Journal & Country Rank*, etc.

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† In Kazan University the Electron Paramagnetic Resonance (EPR) was discovered by Zavoisky E.K. in 1944.

Fractals and fractional integrals: Are there some accurate relationships between them?

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(Received April 1, 2024; accepted April 15, 2024; published April 19, 2024)

This study establishes precise links between the fractional integral of the RL-type and the averaging technique of a smooth function over 1D-fractal sets. These findings were previously reported in the works [1], [2]. To draw in the interest of other experts operating in the NMR/EPR zones, it is helpful to repeat them again. The physical meaning of these acquired formulas is explained and numerical verifications are performed with the purpose of confirming the analytical results. Furthermore, results were achieved for a combination of fractal circuits with a discrete set of fractal dimensions that were generalized. We suppose that these new results help understand deeper the intimate links between fractals and fractional integrals of different types, especially in applications of the fractional operators in complex systems.

PACS: 02.50.-r, 02.60.-x, 02.70.Rr, 05.40.Ca.

Keywords: 1D-Fractals, Riemann-Liouville fractional integrals, averaging procedure over 1D Cantor sets; the generalized self-similar electric circuits

This paper (as mini-review) I decided to devote to the 90-th anniversary of my first Teacher Boris Ivanovich Kochelaev

Preface

Let me to explain firstly (for a potential reader) why I greatly appreciated to this Big Scientist in establishing of my scientific carrier? I should remind of some key points that I wrote 5 years ago because they are really important for me.

When I was the PhD student (1970-1973) I was left by my teacher with the problem alone, because he was invited to spent his "sabbatical leave" in the USA again. This time was very difficult from one side and was instructive for me from another side.

After receiving a red diploma from Kazan State University's Physical Faculty and enrolling in a PhD program, I felt compelled to demonstrate to myself that I could handle challenging problems on my own. I worked for two years to find the right expressions for a relaxation problem in liquid $He^{3,4}$ combinations. But another researcher from Krasnoyarsk Institute of Physics reached exactly the same result using Zubarev's NSO technique, so I was compelled to examine a different topic when it came to the relaxation processes in paramagnets at low temperatures. I resolved another issue in a year and earned the required degree in 1974.

These "testimony" years will always stick in my memory, and I would want to thank my teacher for the "lesson" I learned during that period. These years have given me confidence that I can work independently and find solutions to the issues I face in life. I currently work at the Technical University (KNRTU-KAI) where I take part in problem-solving activities related to engineering. I see them as a fresh challenge that puts my skills and knowledge to the test. Since my Hirsch index is higher than 30 and my works have had more than 5000 citations, I may consider myself as a successful scientist (for more information, see `http://expertcorps.ru/science/whoiswho/ci86`).

A few remarks regarding the original paper that I would like to make to the interested reader. I have authored almost 300 publications, and every one of them that has been published is regarded as my "scientific child". While some of them were fortunate enough to receive good fortune, others were temporarily forgotten. I see myself as an "armor" who suggests fresh approaches to tackle challenging issues that must be "defeated" with the aid of the suggested "new methods". Nevertheless, despite my current situation, I continue to work actively in three "hot" areas: (a) dielectric spectroscopy; (b) fractional calculus and fractals; and (c) the creation of novel statistical techniques for the extraction of deterministic information from random signals.

The "promising child" (b) is the owner of the original document that is displayed below. It creates a clear path between temporal fractional integrals of the Riemann-Liouville type and 1D fractals (as Cantor sets and their generalizations). One of the "hot" problems in the physics of fractional calculus is still this one. It is my sincere hope that this important research will also catch the interest of experts in NMR/EPR applications, particularly in the identification of processes that are represented by fractional integrals/derivatives and are likely now impacted by the presence of a "noise".

1. Introduction

The term FDA, which stands for Fractional Derivative and its Application, has now spread widely. The "hot spot" emerged at the end of the 1980s when numerous researchers from a variety of application fields realized that the new tool provided by fractional calculus mathematics could reveal new features and generalize previously studied fractal geometry phenomena. Some monographs [3–7] and reviews [8, 9] that incorporate extensive old and recent historical surveys and explain the roots of this "hot spot" are recommended for novices. The connection between fractional calculus and fractals is once again of interest. Papers [10–12] described some novel approaches (although without a proper physical explanation). The determination of the justified and correct relationships between the smoothed functions averaged over fractal objects and fractional operators is one of the fundamental issues in the fractional calculus community that has not yet been accurately answered. In monograph [13] and paper [1], this problem was partially solved for the time-dependent functions averaged over Cantor sets, where the influence of an unknown log-periodic function was taken into consideration. This ultimately allowed for the interpretation of the fractional integral with the complex-conjugated power-law exponents. In monograph [13], potential generalizations that could aid in comprehending the function of a spatial fractional integral as a mathematical operator in place of the operation of averaging the smoothed functions over fractal objects were also taken into consideration.

However, in order to receive as a generalization, the desired expressions for the gradient, divergence and curl expressed by means of the fractional operator in the limits of mesoscale (when the current scale η lies in the interval $(\lambda < \eta < \Lambda)$ determining the limits of a possible self-similarity) it was necessary to apply the additional averaging procedure over possible places of location of the fractal object considered. This procedure provides the correct convergence of the microscopic function $f(z)$ at small $(\eta \simeq \lambda)$ and large $(\eta \simeq \Lambda)$ scales. However, the primary mathematical barrier to precisely determining the intended link between the fractal object and the associated fractional integral is the lack of the 2D- and 3D-Laplace transformations. As a result, the fundamental issue this paper addresses can be stated as follows: If we wish to carry out this process without using any approximations, what precise form of the fractional operator is produced in the outcomes of the smoothed function averaging operation across the specified fractal set?

We should also highlight the findings from the most recent publication [14], which presents an original attempt to tie a fractal object – a branching liquid flow stream passing through a porous medium – with the fractional integral. Nevertheless, the effective velocity definition in this model, which is averaged over the fractal's discrete structure, prevents the observation of the normal log-periodic oscillations that are inherent to any discrete-structured fractal object.

In this paper, we want to discuss some new ideas that can help to understand deeper the desired relationship between the accurate averaging of the smoothed functions over fractal sets and fractional integrals in time and space. These new results we consider as a natural generalization of the previous and approximate results achieved in papers [1, 13, 15].

2. New exact relationships connecting 1D fractals and fractional integrals

2.1. The exact relationship between temporal fractional integral and the smoothed function averaged over the Cantor set with M bars.

In paper [13] the relationship between fractional integral with complex power-law exponent and Cantor set has been established. But this relationship was *approximate* and obtained in one-mode approximation and it would be desirable to establish the *exact* relationship between 1D fractals and fractional integrals in time-domain. Attentive analysis shows that one important point in the previous results leading to the desired exact relationship was missed. In order to show it let us reproduce some mathematical expressions that will be helpful for further manipulations and understanding the problem posed. As it has been shown in [13] the Laplace image of the kernel of the Cantor set is described by expression

$$\lim_{N\to\infty} K^{(N)}(z) \equiv K_\nu(z) = \frac{\pi_\nu(\ln(z))}{z^\nu}, \tag{1}$$

where $z = pT(1-x)$, p defines the Laplace parameter, T – is a period of location of the Cantor set, ξ is the scaling factor. The kernel $K^{(N)}(z)$can be presented as

$$K^{(N)}(z) = \prod_{n=-(N-1)}^{N-1} g(z\xi^n) = \prod_{n=0}^{N-1} g(z\xi^n) \prod_{n=1}^{N-1} g(z\xi^{-n}). \tag{2}$$

Here $g(z)$ describes the structure of the given fractal with asymptotic given below. In particular, the Laplace image of the function $g(z)$ for Cantor set having M bars has the form

$$g(z) = \frac{1}{M} \frac{1-\exp\left(-\dfrac{zM}{M-1}\right)}{1-\exp\left(-\dfrac{z}{M-1}\right)}. \tag{3}$$

In order to satisfy to the functional equation of the type (4) in the limit ($N \gg 1$) for the kernel

$$K(\xi z) = \frac{1}{\bar{g}} K(z), \tag{4}$$

the function $g(z)$ should have the following decompositions for small and large values of z for $\mathrm{Re}(z) \ll 1$

$$g(z) = 1 + c_1 z + c_2 z^2 + \ldots, \tag{5}$$

for $\mathrm{Re}(z) \gg 1$

$$g(z) = \bar{g} + \frac{A_1}{z} + \frac{A_2}{z^2} + \ldots. \tag{6}$$

It is easy to note that for the function $g(z)$ from (3) having M bars in each self-similar stage these requirements are satisfied ($\bar{g} = 1/M$).

The solution of the functional equation (4) has the form (1) with power-law exponent ν equaled [10, 11]

$$\nu = \frac{\ln(\bar{g})}{\ln(\xi)} = \frac{\ln(1/M)}{\ln(\xi)}, \quad 0 < \nu < 1. \tag{7}$$

The log-periodic function $\pi_\nu(\ln z \pm \ln \xi) = \pi_\nu(\ln z)$ with period $\ln(\xi)$ that figures in (1) can be decomposed to the infinite Fourier series [1, 13]

$$\pi_\nu\left(\frac{\ln(z)}{\ln(\xi)}\right) = \sum_{n=-\infty}^{\infty} C_n \exp\left(2\pi ni\frac{\ln(z)}{\ln(\xi)}\right) \equiv C_0 + \sum_{n=1}^{\infty}\left(C_n z^{i\Omega_n} + C_n^* z^{-i\Omega_n}\right). \tag{8}$$

Here $\Omega_n = 2\pi n/\ln\xi$ is a set of frequencies providing a periodicity with $\ln\xi$ of product (4). Taking into account this decomposition one can present the limiting solution of (1) in the form

$$K_\nu(z) = C_0 z^{-\nu} + \sum_{n=1}^{\infty}(C_n z^{-\nu+i\Omega_n} + C_n^* z^{-\nu-i\Omega_n}). \tag{9}$$

Here the real exponent ν is defined by expression (7). The presentation of kernel (2) in the form (9) allows reproducing the previous cases (when the sum in (9) is negligible and or it can be presented in one mode approximation [13]) and finding the desired *exact* relationship. Really, taking into account the well-known relationship [14]

$$(p)^{-a} =: \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}\exp(-pt)\,dt, \quad \mathrm{Re}(a) \geq 0, \tag{10}$$

it is easy to find the desired original for the kernel (9)

$$K_\nu(t) = C_0\frac{t^{\nu-1}}{\Gamma(\nu)} + \sum_{n=1}^{\infty}\left(C_n\frac{t^{\nu-1+i\Omega_n}}{\Gamma(\nu+i\Omega_n)} + C_n^*\frac{t^{\nu-1-i\Omega_n}}{\Gamma(\nu-i\Omega_n)}\right). \tag{11}$$

Here we should take into account that temporal variable t should be dimensionless. Based on the determination of the variable $z = pT(1-\xi)$ (see expression (1)) it is easy to restore the dimension of the constant $t \to t/T(1-\xi)$. Based on the definition of the fractional integral in the form of the Riemann-Liouville (RL) type one can write the desired relationship

$$\int_0^t K(t-\tau)\cdot f(\tau)d\tau = C_0 J^\nu f(t) + \sum_{n=1}^{\infty}\left[C_n J^{\nu+i\Omega_n}f(t) + C_n^* J^{\nu-i\Omega_n}f(t)\right]. \tag{12}$$

Here we define the RL-integral of the complex order as

$$J_0^{\nu\pm i\Omega}f(t) = \frac{1}{\Gamma(\nu\pm i\Omega)}\int_0^t (t-\tau)^{\nu-1\pm i\Omega}f(\tau)d\tau. \tag{13}$$

From the exact relationship (12), it follows that the averaging procedure of a smooth function over the generalized Cantor set (having M bars) is accompanied *always* by an infinite set of fractional integrals having complex-conjugated power-law exponents. Only in the partial case when the contribution of log-periodic function becomes negligible relationship (13) restores the RL fractional integral with real power-law exponent [1, 13]. When the total sum in (11) can be replaced approximately by one term

$$\sum_{n=1}^{\infty}\left(C_n\frac{t^{\nu-1+i\Omega_n}}{\Gamma(\nu+i\Omega_n)} + (C_n)^*\frac{t^{\nu-1-i\Omega_n}}{\Gamma(\nu-i\Omega_n)}\right) \cong C\frac{t^{\nu-1+i\langle\Omega\rangle}}{\Gamma(\nu+i\langle\Omega\rangle)} + C^*\frac{t^{\nu-1-i\langle\Omega\rangle}}{\Gamma(\nu-i\langle\Omega\rangle)}, \tag{14}$$

then we restore the basic result of paper [1] in the so-called one-mode approximation. In relationship (14) $\langle\Omega\rangle$ defines the leading mode, which replaces approximately other modes that figure in the left-hand side of expression (14).

2.2. Numerical verification

For verification purposes one can take time-domain expression for the kernel defined by (11) and the corresponding one-mode approximation expression from (14).

Numerical calculations were realized with the help of the following procedure:

1. Calculation of the right side of expression (14) using one-mode approximation constants for some given M (number of the Cantor columns) and the scaling parameter ξ from [1] (the first 3 rows of Table 1).

2. Calculation of the fitting constants C_n ($n = 1, \ldots, N_m$, where N_m defines the finite number of modes in the corresponding sum (11) by the linear least-square method (LLSM)).

The values of fitting parameters for the given values of M and ξ are collected in last 3 rows of Table 1. One can see that the fitting parameters C_0, A_1, Ω_1, are very close to the corresponding initial ones obtained in one-mode approximation.

Table 1. The basic initial (the first 3 rows) and the fitting parameters (rest rows) obtained in the result of numerical verification of expression (11) (this Table is taken from [15])

M	ξ	ν	C_0	A	$\langle\Omega\rangle$	$\langle n\rangle$
2	0.125	0.3333	0.63	0.0082	3.01161	0.9967
5	0.05	0.5372	0.6117	0.0217	2.09144	0.9972
15	0.0167	0.6614	0.606	0.0353	1.5331	0.9991
M	ξ	ν	C_0	C_1	Ω_1	N_m
2	0.125	0.3333	0.62328	0.00805	3.02157	20
5	0.05	0.5372	0.61139	0.02157	2.09438	20
15	0.0167	0.6614	0.60640	0.03523	1.5346	20

Results of further numerical calculations are collected in figures 1, 2 and 3. There one can see that contribution of each next term in the sum (11) decreases drastically in comparison with the previous one. This observation serves as a good proof of the validity of one-mode approximation approach proposed in [13].

After analyzing the data from this part, we discovered that the fitting error value reaches a plateau as N_m increases, and that higher terms, such as $n = 1, 2, \ldots, N_m$, contribute relatively little to the overall outcome. It is evident that the fitting error is primarily dependent on parameter, which describes the primary harmonics contribution.

2.3. Physical interpretation of this result and possible generalizations

To understand deeper the results from physical point of view it makes sense to start from the simplest physical examples which can clarify the procedure considered above. For this purpose we consider at first the following mechanical problem. How to calculate the total and averaged path $\langle L(t)\rangle$ for some interval of the time t for a set of material points (particles) if every particle is moving in one direction with the constant velocity V coinciding with the points of Cantor set

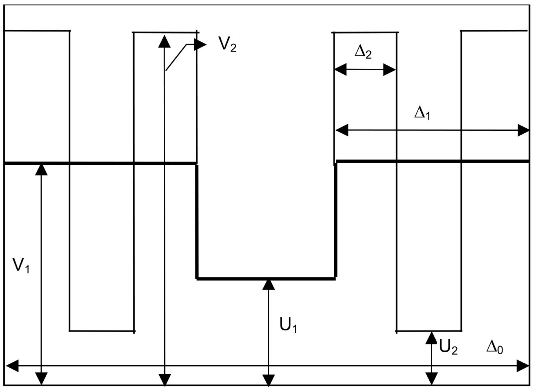

Figure 1. Here two successive stages of the modified binary set are shown. The set of velocities outside of the Cantor set are random. The widths of Cantor bars are denoted as $\{\Delta_i \ (i = 0, 1, 2)\}$, the set of velocities $\{V_i\}$ defines the movement inside the set. (This figure is taken from the book [13])

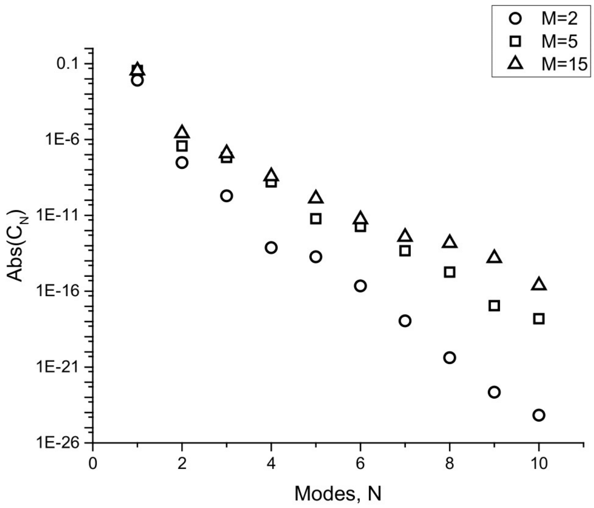

Figure 2. The fitted C_n coefficients presented in log-scale (y-axis) correspond to the successive terms of the sum in (11) (x-axis) for $M = 2$ (open circles), $M = 5$ (open squares) and $M = 15$ (open triangles). (This figure is taken from paper [15])

and is idle outside the set? The expression for the path $L_N(t)$ on the N-th stage of Cantor set construction has the form:

$$L_N(t) = V \int_0^t K_{T,\nu}^{(N)}(\tau) d\tau. \tag{15}$$

Here the value $K_{T,\nu}^{(N)}$ determines the Cantor set located on the temporal interval T on the N-th stage of its self-similarity having M bars and dimension ν defined by expression (7). The value of the normalization interval T for the given case can be found from the condition that during the interval T the body passes the distance L^*. Hence, $T = L^*/V$. Then, integrating expression (11) we obtain

$$L_\nu(t) = \frac{L^*}{\Gamma(1+\nu)} \left(\frac{t}{T}\right)^\nu$$
$$\times \left[1 + \sum_{n=1}^{\infty} \left(c_n \frac{\Gamma(1+\nu)}{\Gamma(\nu+1+i\Omega_n)} \left(\frac{t}{T}\right)^{i\Omega_n} + c_n^* \frac{\Gamma(1+\nu)}{\Gamma(\nu+1-i\Omega_n)} \left(\frac{t}{T}\right)^{-i\Omega_n} \right)\right]. \tag{16}$$

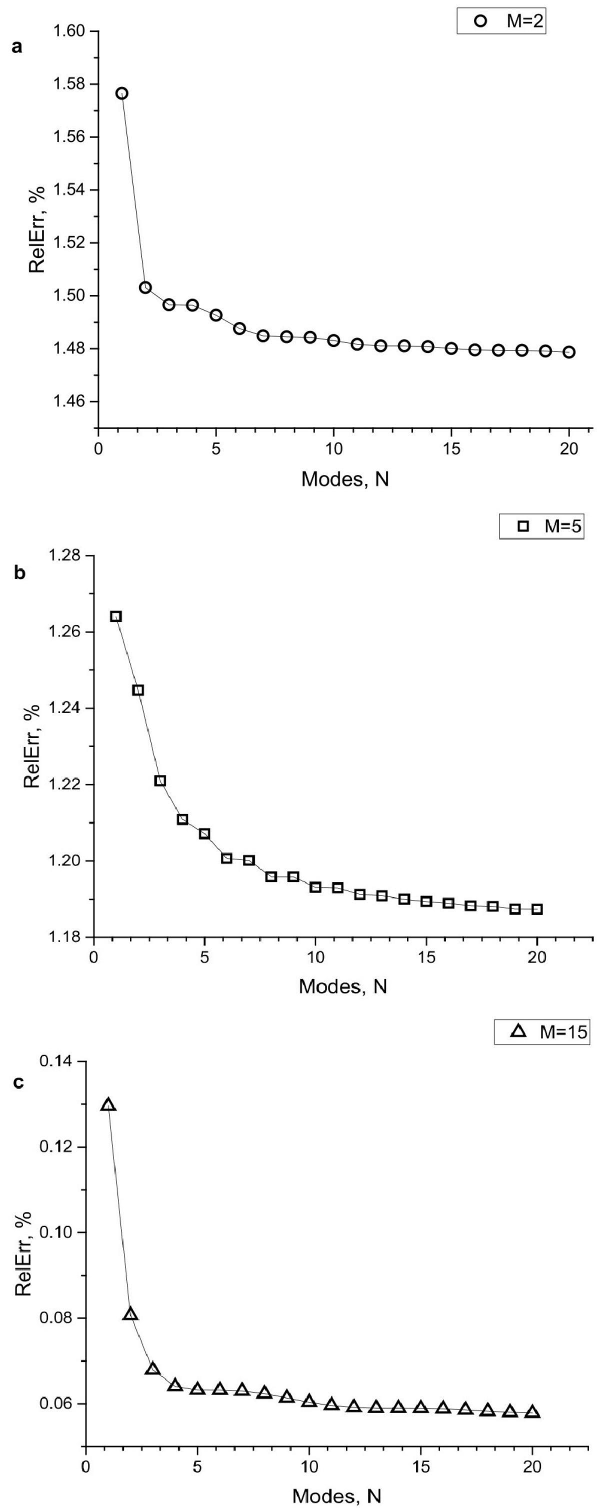

Figure 3. Relative fitting error in % (y-axis) versus number of modes (x-axis) for (a) $M = 2$, (b) $M = 5$ and (c) $M = 15$. (This figure is taken from paper [15])

The sum entering in (16) determines the log-periodic corrections appearing in the result of discretization of the Cantor set. Then, realizing the averaging procedure for the total assembly of particles described in a book [13], we obtain

$$\langle L(t)\rangle = L * \frac{[B(\nu)]}{\Gamma(1+\nu)}\left(\frac{Vt}{L^*}\right)^{\nu}. \qquad (17)$$

Here $B(\nu)$ determines the averaged value of the sum located in the square brackets (16). For the binary Cantor set it can be evaluated analytically that gives

$$B(\nu) = \langle \pi_{T,\nu}(z) \rangle = \int_{-1/2}^{1/2} \pi_{T,\nu}(z + x \ln \xi) dx = \frac{2^{-(1+\nu/2)}}{\ln 2}. \tag{18}$$

In fact, if every particle is just moving inside the Cantor set with the fractal dimension and we are only concerned with the averaged path of the entire assembly, then formula (17) describes the dependence of the averaged path as the function of time for a collection of particles. What portion of the assembly states are involved in this movement is indicated by the fractal dimension ν. It is also important to note that the precise moments at which a particular type of movement transitions into an idle state are averaged. As a result, only a portion of all the states involved in the distribution across Cantor sets are known to us after the averaging method is completed.

The Cantor's "filter", which allows one type of movement to be filtered and another type of movement to be deleted (due to the normalizing of the states) is represented by the averaging operation of a smoothed function over the Cantor set as previously explained. The constant (18) describes the averaged log-periodic states. It is what comes out of this "filtration" process. We examine more complex examples to emphasize the Cantor set's filtration capabilities. Assume for the moment that an assembly of bodies is moved with a range of random velocities, denoted by $\{U_i\}$, in the intervals between Cantor stripes. The steps of Cantor building are indicated by the index $i = 1, 2, \ldots, N, \ldots$. Fig. 1 depicts the updated Cantor set in two steps. The other two figures 2 and 3 demonstrate the influence of the remaining terms for different values of columns M.

It is interesting to set up the following question: *Is there a condition for the distribution of velocities $\{U_i\}$ when the influence of movement outside the set becomes negligible?*

In this case the recurrence relationship for the density is expressed by formula:

$$\begin{gathered} K_{\Delta_N}^{(N)}(t) = V_N \left[K_{\Delta_N}^{(N-1)}(t) + K_{\Delta_N}^{(N-1)}(t - (\Delta_{N-1} - \Delta_N)) \right] \\ +U_N \left[K_{\Delta_N}^{(N-1)}(t - (\Delta_{N-1} - \Delta_N)) - K_{\Delta_N}^{(N-1)}(t - \Delta_N) \right]. \end{gathered} \tag{19}$$

For Laplace-image with the help of retardation theorem

$$F(t - a) =: \exp(-pa) F(p), \tag{20}$$

one can obtain the following expression:

$$\begin{gathered} K_{\Delta_N}^{(N)}(p) = V_N \frac{1 - \exp(-p\Delta_N)}{p} \prod_{n=0}^{N-1} (1 + \exp(-p(\Delta_n - \Delta_{n+1}))) + \\ + \sum_{k=0}^{N-1} U_{k+1} \frac{\exp(-p\Delta_{k+1}) - \exp(-p(\Delta_k - \Delta_{k+1}))}{p} \prod_{n=0}^{k-1} (1 + \exp(-p(\Delta_n - \Delta_{n+1}))). \end{gathered} \tag{21}$$

The total area on the N-th stage is given by the following expression:

$$2^N V_N \Delta_N + \sum_{k=0}^{N-1} 2^k U_{k+1} (\Delta_k - 2\Delta_{k+1}) = S_N. \tag{22}$$

We require that the total area on every stage remains the *same* (it is equivalent to the condition of conservation of the total number of states):

$$S_N = S_{N-1} = \ldots S_0. \tag{23}$$

Let us assume that $U_i = \bar{U} + \delta U_i$, where

$$\bar{U} = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{24}$$

is the arithmetic mean of random velocity. The set $\delta U_i\,(i = 1, 2, \ldots, N, \ldots)$ defines the random deviations of U_i. We assume that these deviations $\{\delta U_i\}$ satisfy the condition:

$$\sum_{k=0}^{N-1} 2^k \delta U_{k+1}(\Delta_k - 2\Delta_{k+1}) = 0. \tag{25}$$

Based on this condition one can rewrite (21) as

$$\begin{aligned} K_{\Delta_N}^{(N)}(p) = \frac{\left(S_0 - \bar{U}T\right)}{(2\xi)^N T}\left[\frac{1-\exp(-pT\xi^N)}{p}\right]\prod_{n=0}^{N-1}\left(1+\exp(-pT(1-\xi)\xi^n)\right) \\ +\bar{U}\frac{1-\exp\left(-pT\right)}{p}. \end{aligned} \tag{26}$$

If we propagate the binary Cantor set on the whole temporal interval ("in" and "out" of the given T) we obtain from

$$K_{\Delta_N}^{(N\gg 1)}(z) \cong \frac{S_0 - \bar{U}T}{T}\prod_{n=-(N-1)}^{N-1}\left[\frac{1+\exp\left(-z\xi^n\right)}{2}\right] + \bar{U}\cdot T\frac{1-\exp\left(-pT\right)}{pT}. \tag{27}$$

The first part of (27) satisfies to the scaling equation (4) with $\bar{g} = 2$ and the second part is proportional to t. If we, again, apply the averaging procedure then we obtain

$$\langle L(t)\rangle = L * \left(1 - \frac{\bar{U}}{V}\right)\frac{B(\nu)}{\Gamma(1+\nu)}\left(\frac{Vt}{L^*}\right)^{\nu} + \bar{U}t. \tag{28}$$

The last expression confirms again the filtration properties of the Cantor set. It divides all motion on two parts: (a) the first motion of the particles located inside the Cantor set, the second part describes the motion that takes place *outside* the fractal set.

2.4. New relationships connecting a fractal process in time with the smoothed function by means of the fractional integral

It is natural to consider another self-similar fractal process, which can be presented in the form of an additive summation [1,13]. These sums are appeared naturally when the self-similar RLC elements connected in parallel or in-series are considered in papers [1,13]

$$S(z) = s_0 \sum_{n=-N}^{N} b^n f\left(z\xi^n\right). \tag{29}$$

Here z is dimensionless Laplace parameter, b and ξ are some constant scaling factors. Each term in (29) can be associated, for example, with the scaled resistor ($R_n = R_0 b^n$), capacitance ($C_n = R/z\xi^n$, $z = j\omega RC$) or inductance ($L_n = Rz\xi^n$, $z = j\omega L_0/R$) that can form a self-similar fractal circuit [13] or with additive contribution of a bar belonging to some Cantor set. In this case, an additive contribution of each bar is expressed by the function

$$f\left(z\xi^n\right) = \frac{1-\exp\left(-z\xi^n\right)}{z\xi^n}. \tag{30}$$

It is easy to note that sum (29) satisfies to the following equation

$$\begin{aligned} S(z\xi) &= s_0 \sum_{n=-N}^{N} b^n f(z\xi^{n+1}) = s_0 \sum_{n=-N+1}^{N+1} b^{n-1} f(z\xi^n) \\ &= \frac{1}{b} S(z) + b^N f\left(z\xi^{N+1}\right) - b^{-N-1} f\left(z\xi^{-N}\right). \end{aligned} \tag{31}$$

We suppose that the contributions of the last two terms on the ends of the finite interval are negligible

$$b^N f\left(z\xi^{N+1}\right) \overset{N\gg 1}{\cong} 0, \quad b^{-N-1} f\left(z\xi^{-N}\right) \overset{N\gg 1}{\cong} 0. \tag{32}$$

Below we will discuss these suppositions in detail. Equation (31) at conditions (32) satisfies the following functional equation that formally coincides with equation (4) considered above

$$S(z\xi) = \frac{1}{b} S(z), \tag{33}$$

So, based on the results obtained in the previous section one can write

$$S(z) = z^{-\nu} \operatorname{Pr}(\ln z), \quad \nu = \ln(b)/\ln(\xi), \quad \operatorname{Pr}(\ln z \pm \ln \xi) = \operatorname{Pr}(\ln z). \tag{34}$$

In spite of the formal coincidence of equation (33) with (4), in order to satisfy to conditions (32) the function $f(z)$ in (31) should have *another* asymptotic decomposition for $\mathrm{Re}(z) \ll 1$

$$f(z) = c_1 z + c_2 z^2 + \ldots, \tag{35}$$

for $\mathrm{Re}(z) \gg 1$

$$f(z) = \frac{A_1}{z} + \frac{A_2}{z^2} + \ldots, \tag{36}$$

and condition

$$1 < b < \xi. \tag{37}$$

As it follows from (20) the last condition provides in addition the obvious inequality

$$0 < \nu = \frac{\ln(b)}{\ln \xi} < 1. \tag{38}$$

So, the results obtained for the sum (31) coincides with the results obtained earlier for the product (2) and the exact relationship in time-domain (11) with subsequent convolution (12) of the kernel (11) with a smooth function remains *invariant*. Is it possible to consider more complicated self-similar process that is presented by an additive combination of two sums of the type?

$$S(z) = s_1 \sum_{n=-N}^{N} b_1^n f_1\left(z\xi^n\right) + s_2 \sum_{n=-N}^{N} b_2^n f_2\left(z\xi^n\right) \equiv S_1(z) + S_2(z). \tag{39}$$

If we assume, by analogy with requirement (32) that the contribution of each sum on the end of the intervals are negligible

$$b_{1,2}^N f_{1,2}\left(z\xi^{N+1}\right) \overset{N\gg 1}{\cong} 0, \quad b_{1,2}^{-N-1} f_{1,2}\left(z\xi^{-N}\right) \overset{N\gg 1}{\cong} 0, \tag{40}$$

(here the number of elements N and the scaling factor ξ for both sums are kept the same values) then from (39) we have a system of equations

$$\begin{aligned} S(z\xi) &= \frac{1}{b_1} S_1(z) + \frac{1}{b_2} S_2(z), \\ S(z\xi^2) &= \frac{1}{b_1^2} S_1(z) + \frac{1}{b_2^2} S_2(z). \end{aligned} \tag{41}$$

Excluding from (39) and the first row of (31) the unknown sums $S_{1,2}(z)$ (defined above by expressions (39)) and inserting them to the second row of (31) we obtain the generalized scaling equation of the second order

$$S(z\xi^2) = \left(\frac{1}{b_1} + \frac{1}{b_2}\right) S\left(z\xi\right) - \frac{1}{b_1 b_2} S(z). \tag{42}$$

Solutions of the scaling equations of the second order was considered in paper [2] and, therefore, we can write

$$S(z) = z^{\nu_1} \Pr_1(\ln z) + z^{\nu_2} \Pr_2(\ln z), \tag{43}$$

where the power-law exponents are found from the equation

$$\xi^{2\nu} - \left(\frac{1}{b_1} + \frac{1}{b_2}\right) \xi^{\nu} + \frac{1}{b_1 b_2} = 0, \quad \nu_{1,2} = \frac{\ln\left(1/b_{1,2}\right)}{\ln \xi}. \tag{44}$$

In solution (43) we have now two log-periodic functions $\Pr_{1,2}\left(\ln z \pm \ln \xi\right) = \Pr_{1,2}(\ln z)$ that can be decomposed to the Fourier series relatively variable $\ln(z)$. Therefore, one can present the solution (43) in the form

$$\begin{aligned} S(z) = C_0^{(1)} z^{-\nu_1} + \sum_{n=1}^{\infty} \left[C_n^{(1)} z^{-\nu_1 + i\Omega_n} + \left(C_n^{(1)}\right)^* z^{-\nu_1 - i\Omega_n}\right] + \\ + C_0^{(2)} z^{-\nu_2} + \sum_{n=1}^{\infty} \left[C_n^{(2)} z^{-\nu_2 + i\Omega_n} + \left(C_n^{(2)}\right)^* z^{-\nu_2 - i\Omega_n}\right], \end{aligned} \tag{45}$$

where we introduced new definitions of the power-law exponents and discrete frequencies reflecting the discrete structure of the fractal process considered.

Moreover,this results can be further generalized. Let us suppose that we have three additive combination of self-similar processes

$$\begin{aligned} S(z) = s_1 \sum_{n=-N}^{N} b_1^n f_1\left(z\xi^n\right) + s_2 \sum_{n=-N}^{N} b_2^n f_2\left(z\xi^n\right) + s_3 \sum_{n=-N}^{N} b_3^n f_3\left(z\xi^n\right) \\ \equiv S_1(z) + S_2(z) + S_3(z). \end{aligned} \tag{46}$$

The functions $f_k(z)$ $(k = 1, 2, 3)$ are chosen in a such way that their contributions on the ends of the interval remain negligible

$$b_k^N f_k\left(z\xi^{N+1}\right) \overset{N \gg 1}{\cong} 0, \quad b_k^{-N-1} f_k\left(z\xi^{-N}\right) \overset{N \gg 1}{\cong} 0, \quad k = 1, 2, 3. \tag{47}$$

We *suppose* again that for each function $f_k(z)$ $(k = 1, 2, 3)$ the asymptotic behavior similar to expressions (35) and (36) is conserved. Therefore, the suppositions (33) are satisfied if the constants b_k and ξ follow the requirements

$$1 < b_k < \xi, \quad k = 1, 2, 3. \tag{48}$$

Therefore, we obtain the following system of equations for the finding of unknown $S_k(z)$

$$\begin{aligned} S(z) &= S_1(z) + S_2(z) + S_3(z), \\ S(z\xi) &= \frac{1}{b_1} S_1(z) + \frac{1}{b_2} S_2(z) + \frac{1}{b_3} S_3(z), \\ S(z\xi^2) &= \frac{1}{b_1^2} S_1(z) + \frac{1}{b_2^2} S_2(z) + \frac{1}{b_3^2} S_3(z). \end{aligned} \tag{49}$$

Excluding from the system (49) the unknown sums $S_k(z)$ and inserting them into the relationship

$$S(z\xi^3) = \frac{1}{b_1^3} S_1(z) + \frac{1}{b_2^3} S_2(z) + \frac{1}{b_3^3} S_3(z), \tag{50}$$

after simple algebraic manipulations we obtain the following functional equation with respect to the function S_z

$$\begin{gathered} S(z\xi^3) = a_2 S\left(z\xi^2\right) + a_1 S\left(z\xi\right) + a_0 S(z), \\ a_2 = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}, \quad a_1 = -\left(\frac{1}{b_1 b_2} + \frac{1}{b_1 b_3} + \frac{1}{b_3 b_2}\right), \quad a_0 = \frac{1}{b_1 b_2 b_3}. \end{gathered} \tag{51}$$

The solution of the functional equation (51) can be written as

$$S(z) = z^{\nu_1} \Pr_1(\ln z) + z^{\nu_2} \Pr_2(\ln z) + z^{\nu_3} \Pr_3(\ln z), \tag{52}$$

The power-law exponents entering into (52) are found from the cubic equation

$$\xi^{3\nu} - a_2 \xi^{2\nu} - a_1 \xi^{\nu} - a_0 = 0, \tag{53}$$

and can expressed as

$$\nu_k = \frac{\ln(1/b_k)}{\ln \xi} = -\frac{\ln(b_k)}{\ln \xi}, \quad 0 < |\nu_k| < 1, \quad k = 1, 2, 3. \tag{54}$$

Therefore, expression (45) conserves its form and for this case the solution (52) can be rewritten as

$$S(z) = \sum_{k=1}^{3} C_0^{(k)} z^{-\nu_k} + \sum_{k=1}^{3} \sum_{n=1}^{\infty} \left[C_n^{(k)} z^{-|\nu_k| + i\Omega_n} + \left(C_n^{(k)}\right)^* z^{-|\nu_k| - i\Omega_n} \right]. \tag{55}$$

It is quite obvious that this result admits the further generalization for the case of $k = 1, 2, \ldots, K$ sums

$$S(z) = \sum_{k=1}^{K} s_k \sum_{n=-N}^{N} b_k^n f_k\left(z\xi^n\right) \equiv \sum_{k=1}^{K} S_k(z). \tag{56}$$

Using the mathematical induction method one can write the final result for this case

$$\begin{gathered} S(z) = \sum_{k=1}^{K} z^{\nu_k} \Pr_k(\ln z) = \\ = \sum_{k=1}^{K} C_0^{(k)} z^{-\nu_k} + \sum_{k=1}^{K} \sum_{n=1}^{\infty} \left[C_n^{(k)} z^{-|\nu_k| + i\Omega_n} + \left(C_n^{(k)}\right)^* z^{-|\nu_k| - i\Omega_n} \right], \end{gathered} \tag{57}$$

The desired roots are found from the relationships

$$|\nu_k| = \frac{\ln(b_k)}{\ln \xi}, \quad 1 < b_k < \xi, \quad 0 < |\nu_k| < 1. \tag{58}$$

So, the kernel connecting the given fractal process with a smooth function in time-domain accept the following general form

$$K_\nu(t) = \sum_{k=1}^{K} C_0^{(k)} \frac{t^{\nu_k - 1}}{\Gamma(\nu_k)} + \sum_{k=1}^{K} \sum_{n=1}^{\infty} \left(C_n^{(k)} \frac{t^{\nu_k - 1 + i\Omega_n}}{\Gamma\left(\nu_k + i\Omega_n\right)} + \left(C_n^{(k)}\right)^* \frac{t^{\nu_k - 1 - i\Omega_n}}{\Gamma\left(\nu_k - i\Omega_n\right)} \right). \tag{59}$$

To the best of our knowledge, this is the most general result that can be obtained for a set of additive self-similar processes in time. As one notice from the calculations, all dynamical functions describe a self-similar process with the *same* scale ξ. If the scales are different and the discrete boundaries of the processes are different also $N_k \neq N$ then these results are becoming questionable and require the further research. But nevertheless, one can show the situation taking place for different scales $\xi_i \neq \xi$ when the result can be reduced to the case considered above. Let us suppose that we have a product combining a random combination of different scales

$$P_n = \prod_{s=1}^{n} \xi_s, \tag{60}$$

but the deviations from some averaged scale (considered as the *dominant* scale) are small. We have for this case

$$P_n = \prod_{s=1}^{n} \xi_s = \prod_{s=1}^{n} (\langle \xi \rangle + \delta_s) = \langle \xi \rangle^n \prod_{s=1}^{n} \left(1 + \frac{\delta_s}{\langle \xi \rangle}\right) = \langle \xi \rangle^n \exp\left(\sum_{s=1}^{n} \ln\left(1 + \frac{\delta_s}{\langle \xi \rangle}\right)\right). \tag{61}$$

For the condition $\delta_s \langle \xi \rangle \ll 1$ we have approximately

$$\begin{gathered} P_n \cong \left(\langle \xi \rangle \exp\left[\frac{\langle \delta \rangle}{\langle \xi \rangle} - \frac{\langle \delta^2 \rangle}{2\langle \xi \rangle^2} + \frac{\langle \delta^3 \rangle}{3\langle \xi \rangle^3}\right]\right)^n, \\ \langle \delta^p \rangle = \frac{1}{n}\sum_{s=1}^{n} \delta_s^p, \quad p = 1, 2, \ldots . \end{gathered} \tag{62}$$

Therefore, this result allows to consider different scales if they are *not* strongly deviated from each other. All the previous results are valid if we simply replace

$$\xi \to \langle \xi \rangle \exp\left(\frac{\langle \delta \rangle}{\langle \xi \rangle}\right) \exp\left(-\frac{\langle \delta^2 \rangle}{2\langle \xi \rangle^2}\right) \ldots, \quad \text{with} \quad \langle \xi \rangle = \frac{1}{n}\sum_{s=1}^{n} \xi_s. \tag{63}$$

3. Results and discussion

Many researchers in the fields of fractional calculus and fractal geometry are still "hot" on the precise links between fractals and fractional integrals. Based on the findings presented in this study, it is evident that the fractional power-law exponent's complex-conjugated component is significant. It is important to note that while the power-law exponent with complex additive has been reported in a few articles, its physical and geometrical origin has not been established. Based on the aforementioned findings, it can be concluded that the complex-conjugated component is closely linked to the fractal process's discrete structure and should, at minimum, be considered in fractional and kinetic equations that attempt to describe self-similar processes in the time domain. As for the spatial fractional integral the finding of the accurate relationship for the given fractal in space remains an open problem. This problem can be divided at least on two parts:

(a) The finding of the proper fractional integral based on the given fractal structure

(b) To find a proper fractal for the fractional integral that is chosen for description of the self-similar process in space.

From our point of view, the general solution of this complex problem is absent because each fractal in space can generate a specific fractional integral [2] but any efforts of researches actively working in this interesting field are very welcome [16, 17].

Acknowledgments

Dr. Prof. Len Dissado and Dr. Prof. Robert Hill, two English researchers, provided significant inspiration for the substance of this study. Profound conversations and seminars with them during the author's (in 1982-1983 years) visit to Chelsea College (London University) enabled the author of this study to comprehend this issue and formulate some problems in this interesting area of research.

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