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† In Kazan University the Electron Paramagnetic Resonance (EPR) was discovered by Zavoisky E.K. in 1944.

‡ Dedicated to Professor Boris Z. Malkin on the occasion of his 85th birthday

About the power of infinite sets†

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Two main concepts of infinity (Aristotle's and Cantor's) are known in the history of mathematics. The last one, prevailing at present, was formulated by founder of the set theory Cantor about a century and a half ago. Cantor used (1) *the diagonal method* to compare the powers of the set of infinite rows of digits 0 and 1 and natural number series; (2) the *Cantor's theorem* about prevalence of the power of the set of all subsets of a set A over the power of A: $|P(\mathrm{A})| > |\mathrm{A}|$. In this work it is shown by use of specific examples that Cantor's reasons can't be considered as strict proofs. Therefore, the concept of the common *potential*(Aristotelian) infinity seems to be more acceptable.

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Millions of years will pass before we'll be able
to understand why we tend to cognize infinity.
P. Erdös.

1. Introduction

The mathematics is rested on three pillars: zero, unit and infinity symbolized respectively as 0, 1 and ∞. These notions are very capacious, intimately interrelated and go out far beyond the mathematics. The unit may be enlarged or be subdivided as much as desired. It is a seed of the natural number sequence emerging as a result of successive summation of unit with itself. The notion of ***mathematical infinity*** is used to symbolize the ***potential***(***virtual***) ***result*** of this unlimited process. Ciphers 0 and 1 are enough to write down any natural number in binary number system:

$$a = a_0 2^0 + a_1 2^1 + \ldots + a_i 2^i + \ldots \equiv a_0 a_1 \ldots a_i \ldots \tag{1}$$

and any proper fraction as dyadic expansion:

$$0.a \equiv 0.a_0 a_1 a_2 \ldots a_n \ldots = a_0 \frac{1}{2^1} + a_1 \frac{1}{2^2} + a_2 \frac{1}{2^3} + \ldots + a_{n-1} \frac{1}{2^n} + \ldots, \tag{2}$$

where each a_n is equal to 0 or 1. For finite natural numbers the series (1) terminates at some finite index n, that is $a_n = 1$, $a_{n+1} = a_{n+2} = \ldots = 0$. The corresponding finite fraction $0.a$ may be represented as "true infinite" fraction according to the rule:

$$0.a = 0.a_0 a_1 \ldots a_{n-1} 1000 \ldots \equiv 0.a_0 a_1 \ldots a_{n-1} 0111 \ldots \tag{3}$$

Each fraction may be considered as a point of the segment $[0, 1]$, and the set of all fractions has a power of the continuum c in the present-day treatment.

The notion *infinity*, widely used also outside the mathematics, is an item of perpetual discussions in the scientific community and also within the general public. In the science it was

†This paper is dedicated to Professor Boris Z. Malkin, who made a significant contribution to the field of magnetic radio spectroscopy in Kazan University, on the occasion of his 85th birthday.

thought since Aristotelian times that "infinity is always in the possibility and not in the actuality". K. Gauss wrote in 1831: "I am against using of infinity value as something completed; that is inadmissible in mathematics". The tradition was violated by the founder of the set theory G. Cantor, who introduced into custom the notion of the completed, "actual" infinity as opposed to the Aristotelian "potential infinity". Cantor generalized the notion of the ***number of elements*** to the infinite sets by using the term ***power*** (or ***cardinal number***) of the set and the symbol $\aleph$ (*aleph*) to represent powers of the infinite sets. According to Cantor ***there exist*** infinite sets with different cardinal numbers. The sequence of the cardinal numbers of the infinite sets in the order of their growth looks like

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots, \tag{4}$$

where the minimal cardinal number $\aleph_0$ is a power of the natural number sequence $\mathbf{N}$. The series (4) must contain the power of continuum c, which exceeds $\aleph_0$: $c > \aleph_0$, as was shown by Cantor using his famous "diagonal" procedure. The assumption $c = \aleph_1$ is called the ***continuum hypothesis***.

Such radical intrusion to the foundations of mathematics met rather ambiguous responses from the prominent scientists which worked at the end of XIX – beginning of XX centuries. Some contradictions (paradoxes) of the ***Cantor's set theory*** were pointed out, and they were thought to be resolvable by the improvement of the logic of the mathematical reasoning. The efforts in this direction resulted in the foundation and development of ***mathematical logic***. The proposed aim of this mathematical branch was the complete formalization of the processes of inference and proof (***Hilbert's program***), which would allow to avoid errors and contradictions. The essential restrictions on the feasibility of this program were imposed by famous ***Gödel's theorems*** about consistency and incompleteness of different systems of axioms. Many contradictions of intuitive Cantor theory fall away within the limits of the modern axiomatic set theories; rather, they were shaded by the choice of axioms. However, some aspects of these theories leave nontransparent as, for instance, the system of alephs (4), which was characterized by H. Weyl as "the mist on the mist". It is not clear weather the continuum hypothesis or its broadened variant are correct.

The above-mentioned relation $c > \aleph_0$ takes an important position in a theory of the cardinal numbers. In this work we present some reasons in favor of alternative relation, $c = \aleph_0$, that is, the power of continuum coincides with the power of the natural number sequence. As a matter of fact, this is the return from the ***Cantor's concept*** of infinity to ***Aristotelian***, or rather the erasing the boundary between them.

Some support in favor of the unique infinity is provided by ***Cantor's very definition*** of the set as "a collection of ***definite, distinguishable objects*** of our perception or our thought conceived as a whole". The collection of "distinguishable" objects may in principle be dissociated into elements and renumbered, that is, this set is ***denumerable*** (finite or countable). Or, in other words, for any two infinite Cantor's sets one may initiate a ***potentially infinite*** process of composing of pairs of elements resulting in the ***one-to-one*** ("1 – 1") ***correspondence*** of these two sets.

Not every collection of objects satisfies Cantor's definition of sets. For instance, since the creation of quantum mechanics the sets of identical particles which obey the quantum-mechanical ***principle of indistinguishability*** have come into the scientific practice. Some restrictions on the quantitative characteristics of the quantum-mechanical sets were also imposed by Heisenberg's principle of uncertainty. In quantum theory the notion of ***Observer*** is explicitly introduced, who regulates the measurement processes, and this is in harmony with inclusion of human ***perception and thought*** into the definition of sets.

The seeming paradoxicality of the equality of continuum and natural number series powers is evidently connected with the representation of these sets as a continuous segment of the number axis and isolated points on this axis correspondingly. However, if one performs the following ***potentially infinite*** procedure on the segment: to throw out the middle one third of the initial segment, then the middle thirds of two remained parts of the initial segment and so on, then as a "result" the so-called triadic set of Cantor (fractal known as *Cantor's dust*) appears. An ***appearance*** of this set is like an infinite collection of points, but it has a power of continuum.

2. Discussion of Cantor's "diagonal procedure"

Two sets are said to be equivalent (or have the same power) if the one-to-one ("$1-1$") correspondence between their elements may be established. Intuitively, the process of pairing the elements will never end if both sets are infinite. In Cantor's theory there are many levels of infinity; this was proved by use of (1) the ***"diagonal procedure"*** and (2) ***the Cantor's theorem***: a power of any set M is less than a power of the set $P(\mathrm{M})$ of all subsets of M.

Let us consider in more detail one of the versions of the diagonal method. Imagine that we can write out all dyadic fractions A in an arbitrary order:

$$\begin{array}{ll} \mathrm{A}_1 & 0.\mathbf{a}_{11}a_{12}a_{13}\ldots a_{1i}\ldots \\ \mathrm{A}_2 & 0.a_{21}\mathbf{a}_{22}a_{23}\ldots a_{2i}\ldots \\ \mathrm{A}_3 & 0.a_{21}a_{22}\mathbf{a}_{23}\ldots a_{2i}\ldots \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \mathrm{A}_i & 0.a_{21}a_{22}a_{23}\ldots \mathbf{a}_{2i}\ldots \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \tag{5}$$

Here each element $a_{ik} = 0$ or 1. The ***diagonal*** of the table (5) arising in the process is $0.a_{11}a_{22}a_{33}\ldots a_{ii}\ldots$. Then we change a_{11} for $a'_{11} = 1 - a_{11}$, a_{22} for $a'_{22} = 1 - a_{22}$ and so on. The resulting fraction

$$\mathrm{A}' = 0.a'_{11}a'_{22}a'_{33}\ldots a'_{ii}\ldots \tag{6}$$

differs from each of the fractions A_i in the table by at least one digit ($a'_{ii} \neq a_{ii}$), that is, this fraction is left without number. Cantor concludes from here that the amount of fractions exceed the amount of numbers, so the set of fractions is not countable and its power is greater than that of the natural series.

However, it seems more natural to assume the diagonal procedure as an additional evidence for the infiniteness of the numeration process. The supposition that ***all*** fractions are written out in table (5) appears to be contradictory. During the process of pairing for establishing the "$1-1$" correspondence neither the fractions nor numbers come to an end.

Note also that the composition of the fraction A' (6) serves as a peculiar application of the axiom of choice (apple of discord in mathematics) – each element of A' is a "reverse" of one element of any function A_i.

The properties of the diagonal method are manifested more brightly if the table of dyadic fractions is written in “natural”, not in arbitrary, order:

$$\begin{array}{ll} f_0 & 0.\mathbf{0}000\ldots0\ldots \\ f_1 & 0.1\mathbf{0}00\ldots0\ldots \\ f_2 & 0.01\mathbf{0}0\ldots0\ldots \\ f_3 & 0.110\mathbf{0}\ldots0\ldots \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ f_n & 0.n_0n_1n_2n_3\ldots\mathbf{0}\ldots0 \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \tag{7}$$

Here $n_0n_1n_2n_3\ldots$ is a number n in dyadic representation:

$$n = n_0n_1n_2n_3\ldots = n_02^0 + n_12^1 + n_22^2 + n_32^3 + \ldots. \tag{8}$$

Certainly, $n_n = 0$, otherwise the term $1\cdot 2^n > n$ would be contained in the sum. The diagonal of the table is $0.000\ldots0\ldots$, so the “absent in the table fraction” is $0.111\ldots1\ldots$ with unrestricted number of ones. Such “fraction” is a point which may be classified as an integer number 1.

Now we have no weighty grounds to assert that there are other fractions which are not presented in the appropriate table. Each fraction A$= 0.a_0a_1a_2a_3\ldots a_i\ldots$ (it may be $\pi - 3$, for which N $\sim 10^{15}$ digits are known) is a limit of the Cauchy sequence

$$0.a_0,\ 0.a_0a_1,\ 0.a_0a_1a_2,\ldots,0.a_0a_1a_2\ldots a_i\ldots.$$

All elements of this sequence (including yet not calculated in the case of $\pi - 3$) are contained in the table, which is guaranteed by the chosen manner of enumeration of fractions.

3. Notes on the Cantor's theorem.

The given above numeration of dyadic fractions table (7) may be used also for the criticism of the Cantor's theorem. Number n of the fraction f_n corresponds to dyadic recording (8) of the number n. This number is uniquely defined by the collection of digits n_i which are equal to 1:

$$n = \sum_{i(n_i=1)} 2^{i-1}. \tag{9}$$

Then the set of indices i, which is a subset of natural series N, may be put into “$1 - 1$” correspondence with the natural number n:

$$n \leftrightarrow \{i - 1 | n_i = 1\}. \tag{10}$$

For instance,

$$1 = 2^0 \leftrightarrow \{0\},\ 2 \leftrightarrow \{1\},\ 27 = 2^0 + 2^1 + 2^3 + 2^4 \leftrightarrow \{0,1,3,4\},$$
$$\text{or yet: } \{a,b,c,\ldots\} \leftrightarrow 2^a + 2^b + 2^c + \ldots.$$

It is evident that in the established “$1 - 1$” correspondence N $\leftrightarrow P(\mathrm{N})$ the number n is not contained in the corresponding subset. Just the subsets, containing a number to which they correspond, play principal role in standard proofs of the Cantor's theorem.

4. Alternative ordering of the natural series

Let us consider one more reason to show that Cantor's theorem can't be applied to infinite sets. It is well known that natural series N have many different order types. There is a lot of ways to present N as a union of nonintersecting infinite sets of numbers. The simplest one is the decomposition to even and uneven numbers:

$$\mathrm{N} = \{1, 3, 5, \ldots; 2, 4, 6, \ldots\} = \{2n - 1\} + \{2n\}.$$

(The sign + stands here for the union of sets.) The decomposition may be continued in an analogous way:

$$\mathrm{N} = \{1, 5, 9, \ldots, 3, 7, 11, \ldots; 2, 6, 10, \ldots, 4, 8, 12, \ldots\} = \{4n - 3\} + \{4n - 1\} + \{4n - 2\} + \{4n\}.$$

So, it is clear that the decomposition of N into nonintersecting subsets can be realized by infinitely many ways. Furthermore N is inexhaustible source of sequences $\{f(n)\}$ where $f(n)$ is an integer-valued growing function as, for instance, n^2, 10^n, $n!$.

For our purposes the following way of decomposition of N seems to be useful. According to the basic theorem of arithmetic each $n \in$N may be uniquely written as a product of powers of the primes p_i:

$$n = p_1^{n_1} p_2^{n_2} \ldots p_i^{n_i} \ldots,$$

so that

$$\mathrm{N} = \sum_{n_1 n_2 n_3 \ldots} p_1^{n_1} p_2^{n_2} \ldots p_i^{n_i} \ldots \tag{11}$$

Here $n_i = 0, 1, 2, 3, \ldots$ and so:

$$\mathrm{N} = 1 + \sum_{m,i} p_i^m + \sum_{mk,ij} p_i^m p_j^k + \sum_{mkl,ijh} p_i^m p_j^k p_h^l + \ldots. \tag{12}$$

If one lets n_i in sum (11) to run also over negative integers, then the set Q of all rational numbers will arise instead of N.

Let P_i be a set of all powers of prime p_i: $\mathrm{P}_i = \{p_i, p_i^2, p_i^3, \ldots\}$. $\mathrm{P} = \{\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \ldots\}$ is a collection of all such sets. The power set of this collection may be presented as follows:

$$\operatorname{Exp} \mathrm{P} = \{\emptyset, \{\mathrm{P}_i\}, \{\mathrm{P}_i \times \mathrm{P}_j\}, \{\mathrm{P}_i \times \mathrm{P}_j \times \mathrm{P}_k\}, \ldots, \{\mathrm{P}_i \times \mathrm{P}_j \times \mathrm{P}_k \ldots\}, \ldots\}. \tag{13}$$

Each element of Exp P corresponds with one of the sums in eq. (12), so the equivalence relation may be established:

$$\operatorname{Exp} \mathrm{P} \sim \mathrm{N}. \tag{14}$$

The power of P (and powers of its constituents P_i) is evidently equal to the power of N, that is, $\aleph_0$. Consequently $|\operatorname{Exp} \mathrm{P}| = 2^{\aleph_0} = \aleph_0$ in accordance with the concept of unique infinity.

5. Conclusion.

As it was mentioned above, the introduction of the set theory into mathematics as its foundation was accompanied by lively discussions. The enthusiastic comments of D. Hilbert and B. Russell alternate with critical statements by A. Poincare and G. Weyl. Poincare wrote in 1908 that "Later generations will regard set theory as a disease from which one has recovered". The fact, that the set theory along with the Cantor's concept of infinity took the firm position in mathematics, may be accompanied by the remark of J. von Neumann to F. Smith: "Young man, in mathematics you don't understand things. You just get used to them". Or, as M. Plank is expressed in connection with the quantum mechanics: "A scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it." It is natural, however, that with the progress of science the necessity to return to controversial points in its history arises. Impracticability of Hilbert's program revived interest to the theory of infinity.

The mathematics, as A. Einstein wrote in 1921, is due by its origin to the necessity to learn something about the behavior of the really existing things. "Mathematical theorems ... are exact until they refer to actuality." Indeed, the very processes of picking out of objects under study from surroundings and measurement influence somehow the results of investigation. The precision of measurements is always finite. Infinity, infinite precision are attainable only mentally. The natural number series is quite enough for solving the problems connected with quantities great (or small) as much as desirable.

An existence of different levels of infinity, which was actually postulated by Cantor, doesn't look like well-founded according to given above reasons. Discussions about foundations of mathematics produce an emotional background which is well described by the following citation from the essay "Mathematics and Logic" by G. Weyl (1946) which is actual until now:

"From this historical essay the following is quite clear: we all less and less believe to the presence of sufficient grounds of logic and mathematics. As all in the modern world we have our own "crisis". And it lasts already about half century. From outside it is hardly noticeable that this crisis interferes our everyday work; however I myself, for instance, must confess that it leaves the deep mark on all my mathematical work ... It constantly damps enthusiasm and resolution with which I set to my scientific researches".

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